On The Cohomology of Congruence Subgroups

Anja Meyer

University of Manchester

Outline

- Group Cohomology
- Congruence Subgroups
- Motivation
- Cohomology of Congruence Subgroups

Group Cohomology

$H^*(G, M)$

G a group, M a kG-module, group cohomolgy assigns a graded ring

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- $H^*((\mathbb{Z}/3\mathbb{Z})^3, \mathbb{F}_3) = S(x_1, x_2, x_3) \otimes \Lambda(y_1, y_2, y_3)\ |x_i| = 2,\ |y_i| = 1$

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- Varying M, ex: integers, p-adics, modulo prime p,...

Congruence Subgroups Γ

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projection $\pi_n : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/n\mathbb{Z})$

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- Galois/Continuous cohomology
- Modular curves
- Representation theory of elliptic curves

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Sylow-p-subgroups $S_p(n)$

- $S_p(n)$ Sylow-p-subgroup of $SL_2(\mathbb{Z}/p^n\mathbb{Z})$
- Have short exact sequence

$$\Gamma(p)^n \to S_p(n) \to S_p(1)$$

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Look at Layers

We reduce mod p^{n-1}

$$\left\{ \begin{pmatrix} 1+p^{n-1}a & p^{n-1}b \\ p^{n-1}c & 1+p^{n-1}d \end{pmatrix} \right\} = \Gamma(p)^n_{p-1} \to SL_2(\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\gamma_{n-1}} SL_2(\mathbb{Z}/p^{n-1}\mathbb{Z})$$

The cohomology

Theorem

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Theorem

Let the group $\Gamma(p)^n$ be defined via the short exact sequence below:

$$\Gamma(p)^n \rightarrow SL_2(\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\gamma_n} SL_2(\mathbb{Z}/p\mathbb{Z})$$

where γ_n is reduction modulo p. Then $\forall n > 1 \in \mathbb{N}$

$$H^*(\Gamma(p)^n, \mathbb{F}_p) = \Lambda[x_1, ..., x_p] \otimes S[y_1...., y_p]$$

Ingredients of Proof

Recall

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More theory

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More theory

- $\Gamma(p)^n$ is a **powerful pro-p group**
- $\Gamma(p)^n$ is Ω-extendable for all $n > 1$.

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Let $p = 3$.
$H^*(S_3(n), \mathbb{F}_3) = H^*(S_3(2), \mathbb{F}_3)\ \forall n > 2$.

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$H^*(SL_2(\mathbb{Z}/3^n\mathbb{Z}), \mathbb{F}_3) = H^*(SL_2(\mathbb{Z}/3^2\mathbb{Z}), \mathbb{F}_3)\ \forall n > 2.$

Thank you for listening :)